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# An initial semantics for the $\mu$-calculus on trees and Rabin's complementation lemma

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**Abstract**

In this paper we show that the function associated with any closed or nonclosed term of the $\mu$-calculus on trees can be represented by a recognizable set of trees whose nodes are labeled by letters and by sets of variables. Rabin's complementation lemma is an immediate consequence of this result.

## 1. Introduction

There is no need to recall the importance, in logic and computer science, of Rabin's complementation lemma [10]: *for any tree automaton $\mathscr{A}$ there is a tree automaton $\mathscr{A}'$ such that $L(\mathscr{A}')$, the set of infinite trees accepted by $\mathscr{A}'$, is the complement of $L(\mathscr{A})$*, that constitutes the main lemma in the Rabin's proof of decidability of the monadic second-order theory of the full $n$-ary tree.

Rabin's proof of this lemma is usually considered as a hard one, and several other proofs have been offered [3, 5, 6, 2]. All these proofs obey the same pattern. They consider a class $\mathscr{O}$ of objects such that a set of trees $L(O)$ is associated with any object $O$ in this class, and they show that $\mathscr{O}$ has the two following properties:

*Equivalence property*: for any set $T$ of trees, there is an automaton $\mathscr{A}$ such that $T = L(\mathscr{A})$ if and only if there is $O \in \mathscr{O}$ such that $T = L(O)$.

*Complementation property*: for any $O$, there is $O'$ in $\mathscr{O}$ such that $L(O')$ is the complement of $L(O)$.

For Gurevich and Harrington [3] and for Muchnik [5], an object of $\mathscr{O}$ is a two-players game $G$ on trees, together with one player $p \in \{I, II\}$. The set $L(G, p)$ is the set of trees on which player $p$ has a winning finite-state strategy for the game $G$. In

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this case the Equivalence property is

$$\forall \mathscr{A},\ \exists G:\ L(\mathscr{A}) = L(G, I),$$
$$\forall G,\ \exists \mathscr{A}:\ L(\mathscr{A}) = L(G, II),$$

and the Complementation property amounts to saying that for any game on any tree, one of the two players has a winning finite-state strategy (see also the survey of Thomas [11]).

For Muller and Schupp [6], $\mathscr{O}$ is the class of alternating automata, which contains the usual automata, so that the Equivalence property is: for any alternating automaton $\mathscr{B}$ there is an automaton $\mathscr{A}$ such that $L(\mathscr{B}) = L(\mathscr{A})$. The Complementation property is obtained as a consequence of the determinacy of certain games.

For Emerson and Jutla [2], $\mathscr{O}$ is the class of all closed terms of the $\mu$-calculus on trees (cf. [9, 4, 7, 1]), for which the Complementation property is easy: every term $\tau$ can be syntactically transformed into its dual $\tilde{\tau}$ where $L(\tilde{\tau})$ is the complement of $L(\tau)$. On the other hand, Niwiński has proved [8] the Equivalence property for the $\mu$-calculus without intersection, which applies also to non closed $\mu$-terms. But even if $\tau$ does not contain intersection, its dual may contain intersection. Thus, Emerson and Jutla, still using a game-theoretical argument, prove that for every closed $\mu$-term $\tau$ there is an alternating automaton $\mathscr{B}$ of a special form such that $L(\tau) = L(\mathscr{B})$ and the Equivalence property for the $\mu$-calculus becomes a consequence of the Equivalence property for these alternating automata, that they prove by the same kind of arguments as those used for the Equivalence property for games.

In this paper we present a tool that allows to prove directly the Equivalence property for the $\mu$-calculus, without any use of alternating automata or game-theoretical arguments. Indeed it allows to prove a more general Equivalence property which applies also to nonclosed $\mu$-terms and which generalizes Niwiński's Equivalence property. In [7], Niwiński considers the function

$$\tau^{\mathscr{T}}[x_1, x_2, \ldots, x_n] : \mathscr{P}(T)^n \to \mathscr{P}(\mathscr{T})$$

associated with a $\mu$-term $\tau$ whose free variables are $x_1, x_2, \ldots, x_n$ (so that when $\tau$ is closed, $\tau^{\mathscr{T}}[] = L(\tau)$), and shows that this function can be represented by a recognizable set of incomplete infinite trees, whose leaves are labeled by variables in $\{x_1, x_2, \ldots, x_n\}$, that is the "initial" semantics of the $\mu$-term, as explained in [7].

For instance, the $\mu$-term $\tau_1 = a(x, b(y, x))$ defines the function $\tau_1^{\mathscr{T}}[x, y]$ such that

$$\tau_1^{\mathscr{T}}[x, y](T_x, T_y) = \{a(t_1, b(t_2, t_3)) \mid t_1 \in T_x, t_2 \in T_y, t_3 \in T_x\},$$

the $\mu$-term $\tau_2 = a(y, x)$ defines the function $\tau_2^{\mathscr{T}}[x, y]$ such that

$$\tau_2^{\mathscr{T}}[x, y](T_x, T_y) = \{a(t_1, t_2) \mid t_1 \in T_y, t_2 \in T_x\}.$$

Then the function $\tau^{\mathscr{T}}[x, y]$ associated with $\tau = \tau_1 \wedge \tau_2$ is defined by

$$\tau^{\mathscr{T}}[x, y](T_x, T_y) = \{a(t_1, b(t_2, t_3)) \mid t_1 \in T_x, t_1 \in T_y, b(t_2, t_3) \in T_x, t_2 \in T_y, t_3 \in T_x\}.$$

We suggest to represent this function by something like $a(\{x, y\}, \langle b, \{x\}\rangle(y, x))$. In such incomplete trees, leaves are labeled by single variables, as usual, or by sets of variables, but also internal nodes may be labeled by variables, expressing some other constraints that have to be taken into account when substituting trees for variables.

This leads us to consider complete infinite trees whose nodes are labeled by a letter *and* a set of variables, called f-trees, because we use them as a concrete representation of *functions* over trees (such trees have been considered both by Muchnik [5], where variables are called "dead-ends", and by Rabin [10]). We give a proper algebraic meaning to sets of f-trees (called f-sets when they satisfy an additional technical condition) by associating with every f-set $F$ over the variables $\{x_1, x_2, \ldots, x_n\}$, a mapping $F[x_1, x_2, \ldots, x_n]: \mathscr{P}(\mathscr{T})^n \to \mathscr{P}(\mathscr{T})$ and by defining an operation of composition of f-sets denoted by $F[x_1 := F_1, \ldots, x_n := F_n]$.

Then we prove our main result:

(1) for any $\mu$-term $\tau$ there exists an f-set $\tau^{\mathscr{F}}$ such that the function associated with $\tau^{\mathscr{F}}$ is equal to the function associated with $\tau$, and

(2) the f-set $\tau^{\mathscr{F}}$ is a recognizable set of trees.

This proof is done by structural induction on $\tau$, the only difficult step being to prove the following result:

> if $F$ is a recognizable f-set of f-trees, then $\nu x.F$ and $\mu x.F$, the greatest and least fixed points of the equation $G = F[x := G]$, are recognizable.

We claim that all the difficulty of Rabin's lemma is concentrated in this result.

Of course, all methods used so far to prove Rabin's lemma can be used to prove this single result. Indeed it is not difficult to convince oneself that Lemmas 3.2 and 3.4 of Rabin [10] are direct proof of this result.

Beyond their interest in the present proof of Rabin's complementation lemma, we believe that f-sets should be a useful tool to define recognizable functions over sets of trees in the way suggested by Thomas in [12] or to investigate the problem posed in [1] of whether or not the alternation-depth hierarchy of the $\mu$-calculus with intersection is finite.

This paper is organized as follows. In Section 1 we recall some notions and notations about trees. In Section 2 we define the notions of f-trees and f-sets, we define the composition of f-sets and prove some properties of this operation. In Section 3 we define the $\mu$-calculus on trees and show how to associate an f-set with every $\mu$-term. In Section 4, we show how to prove the generalized Equivalence property for the $\mu$-calculus.

## 2. Preliminary notions

### 2.1. Notations

We denote by $\emptyset$ the empty set as well as the unique function defined on the empty set.

For any set $E$, $\mathscr{P}(E)$ is the set of subsets of $E$. If $s$ and $s'$ are two functions from $E$ into $\mathscr{P}(F)$, we write $s \subseteq s'$ iff $\forall e \in E, s(e) \subseteq s'(e)$. If $s: E \to \mathscr{P}(F)$ and if $X$ is a subset of $F$, then $s \cap X$ is the function defined by $(s \cap X)(e) = s(e) \cap X$; similarly, $s - X$ is defined by $(s - X)(e) = s(e) - X$.

For any infinite sequence $u \in E^\omega$ of elements of $E$, we denote by $u(n)$, for $n \geqslant 1$, the $n$th element of this sequence, so that $u = u(1)u(2) \cdots u(n) \cdots$, and by $u[n]$, for $n \geqslant 0$, the sequence $u(1)u(2) \cdots u(n)$ of length $n$. In particular, $u[0] = \varepsilon$.

In all what follows, we consider a finite alphabet $A$.

### 2.2. Trees

Let $E$ be any set. An *infinite full binary tree* (or, simply, a *tree*) over $E$ is a mapping $t : \{l, r\}^* \to E$. We denote by $\mathscr{T}(E)$ the set of all trees over $E$. If $E$ is the fixed set $A$, then $\mathscr{T}(A)$ will be written $\mathscr{T}$.

If $t$ is a tree and if $u \in \{l, r\}^*$, then $t|_u$ is the tree defined by $\forall v \in \{l, r\}^*, t|_u(v) = t(uv)$. In particular, $t|_\varepsilon = t$.

If $a \in A$ anf if $t_1, t_2$ are trees, then $a(t_1, t_2)$ is the tree $t$ defined by $t(\varepsilon) = a, t(lu) = t_1(u), t(ru) = t_2(u)$, so that $t|_l = t_1$ and $t|_r = t_2$. If $T_1$ and $T_2$ are two sets of trees then $a(T_1, T_2) = \{a(t_1, t_2) \mid t_i \in T_i, i = 1, 2\}$.

## 3. F-sets

### 3.1. Definitions

Let $X$ be a finite set of variables. An *f-tree* over $X$ is a pair $\langle t, s \rangle$ where $t \in \mathscr{T}$ is a tree over the alphabet $A$ and $s \in \mathscr{T}(\mathscr{P}(X))$ is a tree over the subsets of $X$.

An *f-set* over $X$ is a set $F$ of f-trees over $X$ that satisfies the condition (F): if $\langle t, s \rangle \in F$ then $\langle t, s' \rangle \in F$, for any $s' \in \mathscr{T}(\mathscr{P}(X))$ such that $s \subseteq s'$. We denote by $\mathscr{F}(X)$ the set of all f-sets over $X$. In particular, $\mathscr{F}(\emptyset)$ can be identified with $\mathscr{P}(\mathscr{T})$.

The set $\mathscr{F}(X)$ ordered by inclusion is a complete lattice whose minimal element is the empty set and the maximal one is $\mathscr{T} \times \mathscr{T}(\mathscr{P}(X))$.

### 3.2. Composition of f-sets

#### 3.2.1. Definition

Let $Y, X_1, \ldots, X_n$ be finite sets of variables, let $F, G_1, \ldots, G_n$ be f-sets, respectively over $Y, X_1, \ldots, X_n$, and let $x_1, \ldots, x_n$ be variables. We denote by

$$F[x_1 := G_1, \ldots, x_n := G_n],$$

the f-set $F'$ over $Z = (Y - \{x_1, \ldots, x_n\}) \cup X_1 \cup \cdots \cup X_n$ defined by $\langle t, s' \rangle \in F'$ iff there exists $\langle t, s \rangle \in F$ such that (i) $s - \{x_1, \ldots, x_n\} \subseteq s'$, and (ii) $\forall i = 1, \ldots, n, \forall u \in \{l, r\}^*$, $x_i \in s(u) \Rightarrow \langle t|_u, s'|_u \cap X_i \rangle \in G_i$.

The fact that $F'$ satisfies the condition (F) is a straightforward consequence of the definition.

In particular, if $F \in \mathscr{F}(\{x_1, \ldots, x_n\})$ and if $G_1, \ldots, G_n$ are in $\mathscr{F}(\emptyset) = \mathscr{P}(\mathscr{T})$, then $F[x_1 := G_1, \ldots, x_n := G_n]$ is in $\mathscr{F}(\emptyset)$, so that $F$ can be seen as a mapping from $\mathscr{P}(\mathscr{T})^n$

into $\mathscr{P}(\mathscr{T})$, when the order of variables is given. To be more precise, if we denote by $\mathbf{x}$ a sequence $x_1, x_2, \ldots, x_n$ of variables in a given order, containing all the variables of $F$, then $F[\mathbf{x}]$ is the mapping from $\mathscr{P}(\mathscr{T})^n$ into $\mathscr{P}(\mathscr{T})$ defined by

$$F[\mathbf{x}](T_1, \ldots, T_n) = F[x_1 := T_1, \ldots, x_n := T_n].$$

*3.2.2. Examples*

**Identity.** For any variable $x$, let $I_x$ be the set

$$\{\langle t,s\rangle \in \mathscr{T} \times \mathscr{T}(\mathscr{P}(\{x\})) \mid x \in s(\varepsilon)\}.$$

It is obviously an f-set which has the following property.

**Proposition 1.** $\forall F \in \mathscr{F}(X), I_x[x := F] = F$.

**Proof.** "$\subseteq$" If $\langle t,s'\rangle$ is in $I_x[x := F]$, there exists $\langle t,s\rangle \in I_x$ such that $s - \{x\} \subseteq s'$ and $x \in s(u) \Rightarrow \langle t|_u, s'|_u \cap X\rangle \in F$. Since $\langle t,s\rangle \in I_x$, $x \in s(\varepsilon)$, thus $\langle t, s' \cap X\rangle \in F$. But $\langle t,s'\rangle$ is an f-tree over $(\{x\} - \{x\}) \cup X = X$, so that $s' \cap X = s'$. Hence, $\langle t,s'\rangle \in F$.

"$\supseteq$" Conversely, let $s$ be defined by

$$\forall u \in \{l,r\}^*, s(u) = \begin{cases} \{x\} & \text{if } u = \varepsilon, \\ \emptyset & \text{otherwise} \end{cases}$$

and let $\langle t,s'\rangle \in F$. We have $\langle t,s\rangle \in I_x, s - \{x\} = \emptyset \subseteq s'$, and if $x \in s(u)$ then $u = \varepsilon$ and $\langle t|_u, s'|_u \cap X\rangle = \langle t, s' \cap X\rangle = \langle t,s'\rangle \in F$, so that $\langle t,s'\rangle \in I_x[x := F]$. $\square$

**Union and intersection.** For two given variables $x_1$ and $x_2$, we define the f-sets $\mathrm{V}$ and $\Lambda$ in $\mathscr{F}(\{x_1,x_2\})$ by

$$\langle t,s\rangle \in \mathrm{V} \Leftrightarrow x_1 \in s(\varepsilon) \text{ or } x_2 \in s(\varepsilon),$$
$$\langle t,s\rangle \in \Lambda \Leftrightarrow x_1 \in s(\varepsilon) \text{ and } x_2 \in s(\varepsilon).$$

It is easy to check the following result

**Proposition 2.** *For* $G_1 \in \mathscr{F}(X_1), G_2 \in \mathscr{F}(X_2)$, *let*

$$G_1' = \{\langle t,s\rangle \in \mathscr{T} \times \mathscr{T}(\mathscr{P}(X_1 \cup X_2)) \mid \langle t, s \cap X_1\rangle \in G_1\} \in \mathscr{F}(X_1 \cup X_2)$$

*and*

$$G_2' = \{\langle t,s\rangle \in \mathscr{T} \times \mathscr{T}(\mathscr{P}(X_1 \cup X_2)) \mid \langle t, s \cap X_2\rangle \in G_2\} \in \mathscr{F}(X_1 \cup X_2).$$

*Then*

$$\mathrm{V}[x_1 := G_1, x_2 := G_2] = G_1' \cup G_2',$$
$$\Lambda[x_1 := G_1, x_2 := G_2] = G_1' \cap G_2'.$$

In particular, for $T_1, T_2 \in \mathscr{P}(\mathscr{T}) = \mathscr{F}(\emptyset)$,

$$\mathrm{V}[x_1 := T_1, x_2 := T_2] = T_1 \cup T_2 \text{ and } \Lambda[x_1 := T_1, x_2 := T_2] = T_1 \cap T_2.$$

**Rooting.** For any letter $a$, let $R_a$ be the element of $\mathscr{F}(\{x_1,x_2\})$ defined by

$$\langle t,s\rangle \in R_a \Leftrightarrow t(\varepsilon)=a, x_1\in s(l), x_2\in s(r).$$

**Proposition 3.** *For* $G_1\in\mathscr{F}(X_1), G_2\in\mathscr{F}(X_2)$,

$$R_a[x_1 := G_1, x_2 := G_2]$$
$$= \{\langle t,s\rangle \in \mathscr{T}\times\mathscr{T}(\mathscr{P}(X_1\cup X_2)) \mid t(\varepsilon)=a, \langle t|_l, s|_l \cap X_1\rangle \in G_1, \langle t|_r, s|_r\cap X_2\rangle \in G_2\}.$$

In particular, for $T_1,T_2\in\mathscr{P}(\mathscr{T})=\mathscr{F}(\emptyset), R_a[x_1 := T_1, x_2 := T_2] = a(T_1,T_2)$.

### 3.3. Some properties of composition

In this section we will use freely the vectorial notation $F[\mathbf{x} := \mathbf{G}]$ as an abbreviation for $F[x_1 := G_1,\dots,x_n := G_n]$.

#### 3.3.1. Monotonicity and continuity

It follows directly from its definition that composition is monotonic for inclusion with respect to any argument.

**Proposition 4.** *Let* $F\subseteq F'\in\mathscr{F}(Y)$, $G_i\subseteq G_i'\in\mathscr{F}(X_i)$. *Then*

$$F[\mathbf{x} := \mathbf{G}]\subseteq F'[\mathbf{x} := \mathbf{G}'].$$

It follows that for any $F\in\mathscr{F}(X)$, the equation $G=F[x := G]$ has a least and a greatest fixed point, both in $\mathscr{F}(X-\{x\})$, denoted, respectively, by $\mu x.F$ and $\nu x.F$.

Moreover, the composition is both sup- and inf-continuous in its first argument.

**Proposition 5.** *Let* $F_i$, *for* $i\in I$, *be a family of f-sets in* $\mathscr{F}(Y)$, *and, for* $j=1,\dots,n$, *let* $G_j\in\mathscr{F}(X_j)$. *Then*

$$\left(\bigcup_{i\in I}F_i\right)[\mathbf{x} := \mathbf{G}] = \bigcup_{i\in I}F_i[\mathbf{x} := \mathbf{G}],$$
$$\left(\bigcap_{i\in I}F_i\right)[\mathbf{x} := \mathbf{G}] = \bigcap_{i\in I}F_i[\mathbf{x} := \mathbf{G}].$$

**Proof.** The case of the union is straightforward from the definition. Let us consider the case of the intersection. Obviously,

$$\left(\bigcap_{i\in I}F_i\right)[\mathbf{x} := \mathbf{G}] \subseteq \bigcap_{i\in I}F_i[\mathbf{x} := \mathbf{G}].$$

Conversely, let $\langle t,s'\rangle$ be in $\bigcap_{i\in I}F_i[\mathbf{x} := \mathbf{G}]$. Then for any $i\in I$, there is $\langle t,s_i\rangle\in F_i$ such that $s_i-\{x_1,\dots,x_n\}\subseteq s'$ and $x_j\in s_i(u)\Rightarrow \langle t|_u, s'|_u\cap X_j\rangle \in G_j$. Let us consider

$s = \bigcup_{i \in I} s_i$. Since $s_i \subseteq s$, by (F), $\langle t,s\rangle \in F_i$, thus $\langle t,s\rangle \in \bigcap_{i\in I} F_i$. Moreover, clearly, $s - \{x_1,\ldots,x_n\} \subseteq s'$. If $x_j \in s(u)$, then $x_j \in s_i(u)$ for some $i$, thus $\langle t|_u, s'|_u \cap X_j\rangle \in G_j$. But $s_i \subseteq s \Rightarrow s_i|_u \cap X_j \subseteq s|_u \cap X_j$, and, by (F), $\langle t|_u, s|_u \cap X_j\rangle \in G_j$. Therefore, $\langle t,s'\rangle \in (\bigcap_{i\in I} F_i)[\mathbf{x} := \mathbf{G}]$. $\square$

### 3.3.2. *Associativity*

Now we prove that the composition is associative in the following sense

**Proposition 6.** *Let $F \in \mathscr{F}(Z)$, let $G_i \in \mathscr{F}(X_i)$ for $i = 1,\ldots,n$, and $H_j \in \mathscr{F}(Y_j)$ for $j = 1,\ldots,k$. Let $x_1,\ldots,x_n,y_1,\ldots,y_k$ be $n+k$ distinct variables. Then $F[\mathbf{x} := \mathbf{G}][\mathbf{y} := \mathbf{H}] = F[\mathbf{x} := \mathbf{G}[\mathbf{y} := \mathbf{H}], \mathbf{y} := \mathbf{H}]$.*

**Proof.** To avoid heavy notations, we prove this result for the case $n = k = 1$. The proof for the general case is quite similar.

Let $F \in \mathscr{F}(Z), G \in \mathscr{F}(X), H \in \mathscr{F}(Y)$. Let us consider

$$\begin{aligned}
&F' = F[x := G] \in \mathscr{F}((Z - \{x\}) \cup X),\\
&K' = F'[y := H] \in \mathscr{F}((Z - \{x,y\}) \cup (X - \{y\}) \cup Y),\\
&G' := G[y := H] \in \mathscr{F}((X - \{y\}) \cup Y) \text{ and}\\
&K = F[x := G', y := H] \in \mathscr{F}((Z - \{x,y\}) \cup (X - \{y\}) \cup Y),
\end{aligned}$$

and let us prove that $K = K'$.

We have $\langle t,\sigma\rangle \in K'$
iff $\exists \langle t,s'\rangle \in F'$ such that $s' - \{y\} \subseteq \sigma$ and $\forall u, y \in s'(u) \Rightarrow \langle t|_u, \sigma|_u \cap Y\rangle \in H$
iff $\exists \langle t,s\rangle \in F, s' \in \mathscr{T}(\mathscr{P}((Z - \{x\}) \cup X))$ such that $s - \{x\} \subseteq s', s' - \{y\} \subseteq \sigma$ and

$$\begin{aligned}
&\forall u, x \in s(u) \Rightarrow \langle t|_u, s'|_u \cap X\rangle \in G,\\
&\forall u, y \in s'(u) \Rightarrow \langle t|_u, \sigma|_u \cap Y\rangle \in H.
\end{aligned}$$

On the other hand, we have $\langle t,\sigma\rangle \in K$ iff $\exists \langle t,r\rangle \in F$ such that $r - \{x,y\} \subseteq \sigma$ and

$$\begin{aligned}
&\forall u, x \in r(u) \Rightarrow \langle t|_u, \sigma|_u \cap ((X - \{y\}) \cup Y)\rangle \in G',\\
&\forall u, y \in r(u) \Rightarrow \langle t|_u, \sigma|_u \cap Y\rangle \in H,
\end{aligned}$$

iff $\exists \langle t,r\rangle \in F$ such that $r - \{x,y\} \subseteq \sigma$ and

$$\begin{aligned}
&\forall u, x \in r(u) \Rightarrow \exists \langle t|_u, r_u\rangle \in G \text{ such that } r_u - \{y\} \subseteq \sigma|_u \cap ((X - \{y\}) \cup Y) \text{ and}\\
&\forall v, y \in r_u(v) \Rightarrow \langle t|_{uv}, \sigma|_{uv} \cap Y\rangle \in H,\\
&\forall u, y \in r(u) \Rightarrow \langle t|_u, \sigma|_u \cap Y\rangle \in H.
\end{aligned}$$

If $\langle t,\sigma\rangle \in K'$, we can prove that $\langle t,\sigma\rangle \in K$ by setting $r = s$ and $r_u = s'|_u \cap X$. We have $r - \{x,y\} = s - \{x,y\} \subseteq s' - \{y\} \subseteq \sigma$, $r_u - \{y\} = s'|_u \cap (X - \{y\}) \subseteq \sigma|_u \cap (X - \{y\})$, and

$$x \in r(u) = s(u) \Rightarrow \langle t|_u, r_u \rangle = \langle t|_u, s'|_u \cap X \rangle \in G,$$
$$y \in r(u) = s(u) \Rightarrow y \in s'(u) \Rightarrow \langle t|_u, \sigma|_u \cap Y \rangle \in H,$$
$$y \in r_u(v) = s'(uv) \cap X \Rightarrow \langle t|_{uv}, \sigma|_{uv} \cap Y \rangle \in H.$$

If $\langle t, \sigma \rangle \in K$, we can prove that $\langle t, \sigma \rangle \in K'$ by setting $s = r$, and $s'$ defined by

$$y \in s'(u) \;\Leftrightarrow\; y \in r(u) \text{ or } \exists v, w \in \{l, r\}^* : u = vw, x \in r(v), y \in r_v(w),$$
$$s'(u) - \{y\} = \sigma(u) \cap ((Z - \{x, y\}) \cup (X - \{y\})).$$

Let us remark that $r_u \subseteq s'|_u \cap X$: for $z \in X - \{y\}$, if $z \in r_u(v)$ then $z \in \sigma|_u \cap (X - \{y\})(v)$, thus $z \in s'|_u(v)$; if $y \in r_u(v)$ then $y \in s'(uv) = s'|_u(v)$; thus, $r_u \subseteq s'|_u$ and we get the result, since $r_u \cap X = r_u$.

We have $s' - \{y\} \subseteq \sigma$ by definition of $s'$. We also have $s - \{x\} \subseteq s'$: if $z \in Z - \{x, y\}$ then $z \in s(u) = r(u) \Rightarrow z \in \sigma(u) \Rightarrow z \in s'(u)$; if $y \in s(u) = r(u)$ then $y \in s'(u)$.

If $x \in s(u) = r(u)$ then $\langle t|_u, r_u \rangle \in G$, and $r_u \subseteq s'|_u \cap X$ implies, by (F), $\langle t|_u, s'|_u \cap X \rangle \in G$.

If $y \in s'(u)$ then either $y \in r(u)$ thus $\langle t|_u, \sigma|_u \cap Y \rangle \in H$ or $u = vw$ with $y \in r_v(w)$ thus $\langle t|_{vw}, \sigma|_{vw} \cap Y \rangle \in H$ □

*3.3.3. Compatibility with fixpoint operators*

The composition is also compatible with fixpoint operators.

**Proposition 7.** *Let $F \in \mathscr{F}(Y)$ and $G_i \in \mathscr{F}(X_i)$, for $i = 1, \ldots, n$. Let $x, x_1, \ldots, x_n$ be distinct variables, such that $x \notin X_1 \cup \cdots \cup X_n$. Then*

$$(\mu x . F)[\mathbf{x} := \mathbf{G}] = \mu x . (F[\mathbf{x} := \mathbf{G}])$$

*and*

$$(\nu x . F)[\mathbf{x} := \mathbf{G}] = \nu x . (F[\mathbf{x} := \mathbf{G}]).$$

**Proof.** Let us consider the case of the least fixed point.

Let $F'$ be $F[\mathbf{x} := \mathbf{G}] \in \mathscr{F}(Z)$ with $Z = (Y - \{x_1, \ldots, x_n\}) \cup X_1 \cup \cdots \cup X_n$. It is well known that $\mu x . F' \in \mathscr{F}(Z - \{x\})$ is equal to $F'_\alpha$, for some ordinal $\alpha$, where $F'_\alpha$ is inductively defined by: $F'_0 = \emptyset$, $F'_{\alpha+1} = F'[x := F'_\alpha]$, $F'_\beta = \bigcup_{\alpha < \beta} F'_\alpha$, for a limit ordinal $\beta$. Similarly, $\mu x . F \in \mathscr{F}(Y - \{x\})$ is equal to $F_\alpha$, defined by: $F_0 = \emptyset$, $F_{\alpha+1} = F[x := F_\alpha]$, $F_\beta = \bigcup_{\alpha < \beta} F_\alpha$, for a limit ordinal $\beta$.

Let us remark that $F'_{\alpha+1} = F[\mathbf{x} := \mathbf{G}][x := F'_\alpha]$, which is equal, by Proposition 6, to $F[x := F'_\alpha, \mathbf{x} := \mathbf{G}[x := F'_\alpha]]$. But $x \notin X_i$ thus $G_i[x := F'_\alpha] = G_i$. It follows that $F'_{\alpha+1} = F[x := F'_\alpha, \mathbf{x} := \mathbf{G}]$.

Since $F'_0 = \emptyset = F_0[\mathbf{x} := \mathbf{G}]$, we can prove by induction, using Propositions 6 and 5, that $F_\alpha[\mathbf{x} := \mathbf{G}] = F'_\alpha$ that proves the result.

The case of the greatest fixed point is quite similar, replacing union by intersection, and taking $F'_0 = \mathscr{T} \times \mathscr{T}(Z - \{x\})$, $F_0 = \mathscr{T} \times \mathscr{T}(Y - \{x\})$. Thus it is sufficient to prove that $F_0' = F_0[\mathbf{x} := \mathbf{G}]$. In one direction, it is immediate that $F_0[\mathbf{x} := \mathbf{G}] \subseteq F_0'$. Let us prove the converse inclusion. Let $\langle t, s'\rangle \in F_0'$. The f-tree $\langle t, s\rangle$, where $s$ is defined by $s(u) = \emptyset$, for each $u \in \{l, r\}^*$, is in $F_0$ and, by definition of composition, $\langle t, s\rangle$ is in $F_0[\mathbf{x} := \mathbf{G}]$ and $\langle t, s\rangle$ too, since $s \subseteq s'$. □

## 4. $\mu$-calculus on trees

### 4.1. Syntax

Let us consider a set $X = \{x_1, x_2, \ldots\}$ of variables and a finite alphabet $A$. The set $\mathscr{M}$ of $\mu$-terms $\tau$ is defined inductively, together with the sets $FV(\tau)$ of free variables of $\tau$ by:

- a variable $x$ is a $\mu$-term and $FV(x) = \{x\}$,
- if $a$ is in $A$, and $\tau_1, \tau_2 \in \mathscr{M}$, then $\tau = a(\tau_1, \tau_2) \in \mathscr{M}$, and $FV(\tau) = FV(\tau_1) \cup FV(\tau_2)$,
- if $\tau_1$ and $\tau_2$ are in $\mathscr{M}$, then $\tau_1 \vee \tau_2$ and $\tau_1 \wedge \tau_2$ are in $\mathscr{M}$, and $FV(\tau_1 \vee \tau_2) = FV(\tau_1 \wedge \tau_2) = FV(\tau_1) \cup FV(\tau_2)$,
- if $\tau \in \mathscr{M}$, and if $x$ is any variable, then $\tau' = \mu x.\tau$ and $\tau'' = \nu x.\tau$ are in $\mathscr{M}$, and $FV(\tau') = FV(\tau'') = FV(\tau) - \{x\}$.

A $\mu$-term $\tau$ is *closed* if $FV(\tau) = \emptyset$.

### 4.2. Semantics

With each $\mu$-term $\tau$, and with any ordering $\mathbf{x} = \langle x_1, \ldots, x_n\rangle$ of a set $X$ of variables containing all the free variables of $\tau$, we associate a monotonic mapping $\tau^{\mathscr{T}}[\mathbf{x}] : \mathscr{P}(\mathscr{T})^n \to \mathscr{P}(\mathscr{T})$ defined by induction on the construction of $\tau$. Let $\mathbf{T} = \langle T_1, T_2, \ldots, T_n\rangle$. Then

- if $\tau = x_i$, then $\tau^{\mathscr{T}}[\mathbf{x}](\mathbf{T}) = T_i$,
- if $\tau = a(\tau_1, \tau_2)$, then $\tau^{\mathscr{T}}[\mathbf{x}](\mathbf{T}) = a(\tau_1^{\mathscr{T}}[\mathbf{x}](\mathbf{T}), \tau_2^{\mathscr{T}}[\mathbf{x}](\mathbf{T}))$,
- $(\tau_1 \vee \tau_2)^{\mathscr{T}}[\mathbf{x}](\mathbf{T}) = \tau_1^{\mathscr{T}}[\mathbf{x}](\mathbf{T}) \cup \tau_2^{\mathscr{T}}[\mathbf{x}](\mathbf{T})$,
  $(\tau_1 \wedge \tau_2)^{\mathscr{T}}[\mathbf{x}](\mathbf{T}) = \tau_1^{\mathscr{T}}[\mathbf{x}](\mathbf{T}) \cap \tau_2^{\mathscr{T}}[\mathbf{x}](\mathbf{T})$,
- $(\mu x.\tau)^{\mathscr{T}}[\mathbf{x}](\mathbf{T})$ and $(\nu x.\tau)^{\mathscr{T}}[\mathbf{x}](\mathbf{T})$ are, respectively, the least and the greatest solution of the equation

$$T = \tau^{\mathscr{T}}[x, \mathbf{x}](T, \mathbf{T}), \text{ if } x \text{ does not occur in } \mathbf{x},$$
$$T = \tau^{\mathscr{T}}[\mathbf{x}](T_1, \ldots, T_{i-1}, T, T_{i+1}, \ldots, T_n), \text{ if } x = x_i \text{ for some } i \in \{1, \ldots, n\}.$$

### 4.3. F-sets and the semantics of $\mu$-terms

Now, with any $\mu$-term $\tau$, we associate an f-set $\tau^{\mathscr{F}} \in \mathscr{F}(X)$, where $X = FV(\tau)$, by induction on the construction of $\tau$, using the f-sets defined in Section 3.2.2.

- if $\tau = x$, then $\tau^{\mathscr{F}} = I_x$,
- if $\tau = a(\tau_1, \tau_2)$, then $\tau^{\mathscr{F}} = R_a[x_1 := \tau_1^{\mathscr{F}}, x_2 := \tau_2^{\mathscr{F}}]$,
- $(\tau_1 \vee \tau_2)^{\mathscr{F}} = \mathrm{V}[x_1 := \tau_1^{\mathscr{F}}, x_2 := \tau_2^{\mathscr{F}}]$, $(\tau_1 \wedge \tau_2)^{\mathscr{F}} = \Lambda[x_1 := \tau_1^{\mathscr{F}}, x_2 := \tau_2^{\mathscr{F}}]$,

- $(\mu x.\tau)^{\mathscr{F}}$ is the least solution $\mu x.\tau^{\mathscr{F}}$ of the equation $G = \tau^{\mathscr{F}}[x := G]$, and $(\nu x.\tau)^{\mathscr{F}}$ is the greatest solution $\nu x.\tau^{\mathscr{F}}$ of this equation, these solutions being taken in $\mathscr{F}(X - \{x\})$.

It is easy to check that $\tau^{\mathscr{F}} \in \mathscr{F}(FV(\tau))$. This f-set $\tau^{\mathscr{F}}$ is a "concrete" representation of the semantics of $\tau$ in the following sense.

**Theorem 1.** *For any ordering* $\mathbf{x}$ *of a set* $X$ *containing* $FV(\tau)$, $\tau^{\mathscr{T}}[\mathbf{x}] = \tau^{\mathscr{F}}[\mathbf{x}]$.

**Proof.** The proof is by induction on the construction of $\tau$.

If $\tau = x_i$ then $\tau^{\mathscr{F}} = I_{x_i}$, and by extending the proof of Proposition 1, $I_{x_i}[\mathbf{x}](\mathbf{T}) = T_i = \tau^{\mathscr{T}}[\mathbf{x}](\mathbf{T})$.

For $\tau = \tau_1 \vee \tau_2$, or $\tau = \tau_1 \wedge \tau_2$, or $\tau = a(\tau_1, \tau_2)$, this follows from the induction hypothesis and the Propositions 6, 2, and 3.

Using the induction hypothesis, it remains to prove that $(\mu x.\tau^{\mathscr{F}})[\mathbf{x}](\mathbf{T})$ and $(\nu x.\tau^{\mathscr{F}})[\mathbf{x}](\mathbf{T})$, are, respectively, the least and the greatest solutions of the equation

$$T = \tau^{\mathscr{F}}[x, \mathbf{x}](T, \mathbf{T}), \text{ if } x \text{ does not occur in } \mathbf{x},$$

$$T = \tau^{\mathscr{F}}[\mathbf{x}](T_1, \dots, T_{i-1}, T, T_{i+1}, \dots, T_n), \text{ if } x = x_i \text{ for some } i \in \{1, \dots, n\},$$

which is a consequence of Proposition 7. □

## 5. Recognizable f-sets

### 5.1. Recognizable sets of trees

A tree automaton over the alphabet $A$ is a tuple $\mathscr{A} = \langle Q, \delta, Q_I, \mathscr{F} \rangle$, where $Q$ is a set of states, $\delta \subseteq Q \times A \times Q \times Q$ is a set of transitions, $Q_I \subseteq Q$ is a set of initial states and $\mathscr{F}$ is a subset of $\mathscr{P}(Q)$.

A *run* of $\mathscr{A}$ on a tree $t \in \mathscr{T}(A)$ is a mapping $\rho : \{l, r\}^* \to Q$ such that $\rho(\varepsilon) \in Q_I$ and $\forall u \in \{l, r\}^*, \langle \rho(u), t(u), \rho(ul), \rho(ur) \rangle \in \delta$.

A run $\rho$ is *accepting* if for any branch $u$ (seen as an element of $\{l, r\}^\omega$), the set of states occurring infinitely often in the infinite sequence $\rho(u[0])\rho(u[1]) \cdots \rho(u[n]) \cdots$ belongs to $\mathscr{F}$.

A tree $t$ is *accepted* by $\mathscr{A}$ if there is an accepting run of $\mathscr{A}$ on $t$. The set of trees accepted by $\mathscr{A}$ is denoted by $L(\mathscr{A})$. A set $T$ of trees is *recognizable* if there is an automaton $A$ such that $T = L(\mathscr{A})$, and we say that $\mathscr{A}$ *recognizes* $T$.

It is well known (cf. [10]), and easy to prove, that the family $\mathrm{Rec}(A)$ of recognizable subsets of $\mathscr{T}(A)$ is closed under union and intersection.

Let us consider a second alphabet $A'$ and a mapping $\pi : A \to A'$. The *projection* of a tree $t \in \mathscr{T}(A)$ is the tree $\pi(t) \in \mathscr{T}(A')$ defined by $\forall u \in \{l, r\}^*, \pi(t)(u) = \pi(t(u))$, and the projection of a set $T$ is $\pi(T) = \{\pi(t) \mid t \in T\}$. The *inverse projection* of $T' \subseteq \mathscr{T}(A')$ is $\pi^{-1}(T') = \{t \in \mathscr{T}(A) \mid \pi(t) \in T'\}$. The family of all recognizable sets of trees is closed under projection and inverse projection.

### 5.2. The complementation lemma

Because of the natural isomorphism $\mathscr{T}(A) \times \mathscr{T}(\mathscr{P}(X)) \simeq \mathscr{T}(A \times \mathscr{P}(X))$, any f-set is a set of trees over the alphabet $A \times \mathscr{P}(X)$.

Then it is clear that, for two given sets $X$ and $Y$ of variables, the mapping $\langle t,s\rangle \rightsquigarrow \langle t,s-Y\rangle$ (resp. $\langle t,s\cap Y\rangle$) is a projection from $\mathscr{T}(A\times\mathscr{P}(X))$ to $\mathscr{T}(A\times\mathscr{P}(X-Y))$ (resp. $\mathscr{T}(A \times \mathscr{P}(X \cap Y))$.

The complementation lemma is a straightforward consequence of the following result.

**Proposition 8.** *Let $\tau$ be a $\mu$-term and let $X = FV(\tau)$ be its set of free variables. The f-set $\tau^{\mathscr{F}} \in \mathscr{F}(X)$ is recognizable as a subset of $\mathscr{T}(A \times \mathscr{P}(X))$.*

The proof is by induction on the construction of $\tau$. The set $I_x$ is obviously recognizable. If $\tau = \tau_1 \vee \tau_2$ (resp. $\tau = \tau_1 \wedge \tau_2$) and if $G_1 = \tau_1^{\mathscr{F}}$ and $G_2 = \tau_2^{\mathscr{F}}$, then $\tau^{\mathscr{F}} = \mathrm{V}[x_1 := G_1, x_2 := G_2]$ (resp.$\tau^{\mathscr{F}} = A[x_1 := G_1, x_2 := G_2]$), and, by Proposition 2, $\tau^{\mathscr{F}}$ is the union (resp. the intersection) of the inverse projections $G_1'$ and $G_2'$ of $G_1$ and $G_2$. Thus, if $G_1$ and $G_2$ are recognizable, so is $\tau^{\mathscr{F}}$. If $\tau = a(\tau_1, \tau_2)$, then, by Proposition 3, $\tau^{\mathscr{F}}$ is the set of all trees $\langle a, Y\rangle(t_1, t_2)$ with $Y \subseteq X$, such that $t_i$ belongs to some inverse projection of $\tau_i^{\mathscr{F}}$. Thus if $\tau_1^{\mathscr{F}}$ and $\tau_2^{\mathscr{F}}$ are recognizable, so is $\tau^{\mathscr{F}}$.

Since $(\mu x \tau)^{\mathscr{F}} = \mu x \,.\, \tau^{\mathscr{F}}$ and $(\nu x \,.\, \tau)^{\mathscr{F}} = \nu x \,.\, \tau^{\mathscr{F}}$, it remains to prove that if an f-set $F$ is recognizable, so are the f-sets $\mu x \,.\, F$ and $\nu x \,.\, F$. As explained in the introduction, there are several ways of proving this result. However, the more direct one seems to construct the automaton $\mathscr{A}_\mu$ and $\mathscr{A}_\nu$ recognizing $\mu x \,.\, F$ and $\nu x \,.\, F$ given an automaton $\mathscr{A}$ recognizing $F$. This construction has been done by Rabin [10, Lemmas 3.2 and 3.4] and is the core of his proof.


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